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# THE NUMERICAL SOLUTION OF THE KALMAN FILTERING PROBLEM

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## 1. INTRODUCTION

In this contribution we consider the discrete linear Kalman filtering problem. We present a short historical perspective of the Kalman filtering problem covering the last 25 years starting with the Kalman paper of 1960. We aim to show what has been achieved and finish by indicating what currently is required for the efficient and reliable numerical solution of the problem.

While the material is not new, the numerical considerations may not be familiar to the control community and so we use these as motivation for the developments. The more recent work of Paige and others does not seem to be well known or its utility fully appreciated, even in the numerical community. Certainly quality software for the Kalman filtering problem has yet to appear and some work is necessary to get to this stage.

We consider the model

$$\begin{aligned} x_k &= A_k x_{k-1} + u_k, \quad E(u_k) = 0, \quad E(u_k u_k^T) = U_k \\ y_k &= C_k x_k + v_k, \quad E(u_k) = 0, \quad E(v_k v_k^T) = V_k \end{aligned} \tag{1.1}$$

where $x_k$ is the n element vector of state variables, $y_k$ the m element vector of measured, or observed, variables, $u_k$ is the vector of process noise and $v_k$ the vector of measurement noise. $u_k$ and $v_k$ are usually assumed to come from a normal distribution. This simple form is quite sufficient for our purposes, but we note that no additional complication ensues if control variables are included in the model. We hope that our use of $u_k$ to denote the process noise rather than the control does not cause too much disturbance to the reader.

The problem we wish to consider is a particular case of the Wiener filtering problem; given observations $y_0, y_1, \ldots, y_k$ find the best linear unbiased estimate, $\hat{x}_k$, of $x_k$. Additional statistical information such as $E(\hat{x}_k \hat{x}_k^T)$ is also frequently required.

Although we shall only look at the filtering problem, the techniques can usually be readily extended to allow smoothing and prediction.

## 2. THE KALMAN FILTER

We now state, without proof, the solution given by Kalman (1960) to the

filtering problem. Background, proof and additional information can be found in the Kalman paper. Since publication of the paper the problem of determining $\hat{x}_k$ has become known generically as the Kalman filtering problem.

Put

$$E(x_k x_k^T) = X_k, \quad E(y_k y_k^T) = Y_k \quad \text{and} \quad E(\hat{x}_k \hat{x}_k^T) = \hat{X}_k. \tag{2.1}$$

Then, to help see the reason for the definitions of $\tilde{X}_k$ and $\tilde{Y}_k$ below, we note that

$$X_k = A_k X_{k-1} A_k^T + U_k \quad \text{and} \quad Y_k = C_k X_k C_k^T + V_k. \tag{2.2}$$

Now define

$$\tilde{x}_k = A_k \hat{x}_{k-1}, \quad \tilde{X}_k = A_k \hat{X}_{k-1} A_k^T + U_k \tag{2.3}$$

and

$$\tilde{Y}_k = C_k \tilde{X}_k C_k^T + V_k, \quad K_k = \tilde{X}_k C_k^T \tilde{Y}_k^{-1}. \tag{2.4}$$

(2.3) is called the time update, $\tilde{x}_k$ the predictor, or estimator, $\tilde{X}_k$ the predicted covariance matrix and $K_k$ the Kalman gain matrix. Kalman showed that

$$\hat{X}_k = \tilde{X}_k - \tilde{X}_k C_k^T \tilde{Y}_k^{-1} C_k \tilde{X}_k, \tag{2.5}$$

and

$$\hat{x}_k = \tilde{x}_k + K_k(y_k - C_k \tilde{x}_k). \tag{2.6}$$

(2.5) and (2.6) are called the measurement update and (2.6) is, of course, the Kalman filter.

Because covariance matrices are recurred this is known as a covariance filter. The matrix $X_k^{-1}$ is called an information matrix and a corresponding information filter can also be developed.

## 3. NUMERICAL DIFFICULTIES WITH THE KALMAN FILTER

Implementing the Kalman filter directly from equations (2.5) and (2.6) can give rise to severe numerical difficulties and in this section we indicate the source of these difficulties. Firstly, there are several computations of the form $B^T Z B$ to be performed. To simplify matters let us take $Z = I$ and consider the computation of $B^T B$. It is well established that, numerically, such computations give rise to a loss of information (Golub, 1965; Hammarling, 1985). Consider the simple example where

$$B = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix} \quad \text{so that} \quad B^T B = \begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon^2 \end{bmatrix}.$$

A perturbation of order $\epsilon$ is required to change the rank of B, whereas it requires only a perturbation of order $\epsilon^2$ to change the rank of $B^TB$. If $\epsilon$ is above noise level, but $\epsilon^2$ is below noise level then, in the presence of such noise, while B has rank two, we cannot say whether $B^TB$ has rank one or two. We lose important information in forming $B^TB$ whenever we are not close to orthogonality and the situation gets worse the closer we are to rank deficiency. We should bear in mind that in many applications, such as signal processing, the data can have quite low accuracy. We shall mention another example of the numerical danger of forming normal matrices in Section 4.

The second difficulty is in (2.5) where the matrix $\tilde{Y}_k^{-1}$ is required, but $\tilde{Y}_k$ may be ill-conditioned or even singular, in which case the computation breaks down completely. In particular if any stage is noise free then $\tilde{Y}_k$ is singular and the common practice of adding in artifical process noise hardly seems a satisfactory means of allowing the computation to proceed. It seems aesthetically displeasing to have a computation whose numerical properties deteriorate with better information.

The developments in Kalman filtering discussed in subsequent sections have occurred to try and avoid one or both of the above problems.

A consequence of these difficulties is that the computed $\hat{X}_k$ may not be positive semi-definite, in which case one may obtain negative variances, or correlations outside the range $[-1,1]$.

Finally we note that poor scaling may exacerbate the above problems and it can be very important to balance the system (1.1) before performing other computations. This is an observation that is generally true of such system models and it is important to consider balancing, even if subsequent computations are numerically stable, otherwise a poorly scaled problem may be transformed into a genuinely ill-conditioned problem (Parlett and Reinsch, 1971; Laub, 1979; Ward, 1981; Williams, 1985).

It seems to be not uncommon for control models to be poorly scaled, usually because the units of measurement chosen for the variables give values of widely different magnitudes.

## 4. THE CHOLESKY AND QR FACTORIZATION

In this section we briefly discuss two important factorizations, namely the Cholesky and QR factorizations. Both these factorizations can be computed by numerically stable methods (see for example Golub and Van Loan, 1983) and reliable numerical software is readily available for both factorizations (Dongarra et al, 1979; IMSL; NAG), so we shall not discuss the algorithmic details here.

The Cholesky factorization of a symmetric positive semi-definite matrix X is given by

$$X = S^TS, \tag{4.1}$$

where S is an upper triangular matrix, which can be chosen to have non-negative diagonal elements, called the Cholesky factor of X. In the control literature S is often called a square root of X.

If B is an m by n matrix with $m \geqslant n$ then the QR factorization of B is given by

$$B = Q \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{4.2}$$

where Q is an m by m orthogonal matrix and R is an n by n upper triangular matrix, which again can be chosen to have non-negative diagonal elements. Now from (4.2)

$$B^TB = (R^T \quad 0)\, Q^TQ \begin{bmatrix} R \\ 0 \end{bmatrix} = R^TR \tag{4.3}$$

and hence R is the Cholesky factor of the matrix $B^TB$. Thus the QR factorization allows us to avoid forming the matrix $B^TB$, we can obtain the Cholesky factor by performing numerically stable operations directly on B.

To illustrate this let us consider the linear least squares problem given by

$$\text{minimize } e^Te, \quad \text{where } b = Bx + e, \tag{4.4}$$

B being an m by n matrix of observations and b an m element vector of observations of the dependent variable. If B has full rank then x is the solution of the normal equations

$$B^TBx = B^Tb. \tag{4.5}$$

If we find the Cholesky factorization of $B^TB$ these equations become

$$R^TRx = B^Tb$$

which can be solved by firstly using forward substitution to solve the lower triangular equations

$$R^Tz = B^Tb \tag{4.6}$$

for z, then using backward substitution to solve the upper triangular equations

$$Rx = z \tag{4.7}$$

for x. If instead we find the QR factorization of B and put

$$Q^Te = r \equiv \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad \text{and} \quad Q^Tb = c \equiv \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{4.8}$$

then, since $e^Te = r^Tr$, (4.4) becomes

$$\text{minimize } r^Tr, \quad \text{where } c = \begin{bmatrix} R \\ 0 \end{bmatrix} x + r \tag{4.9}$$

which gives

$$c_1 = Rx + r_1 \quad \text{and} \quad c_2 = r_2.$$

Since $r_2$ is fixed $r^Tr$ is minimized by the choice $r_1 = 0$. Thus we can obtain x as the solution of the upper triangular equations

$$Rx = c_1 \tag{4.10}$$

for which

$$e^Te = r_2^Tr_2. \tag{4.11}$$

(Golub, 1965; Golub and Van Loan, 1983, Chapter 6.) Thus, we can solve the least squares problem without having to take the numerically damaging step of forming $B^TB$. Further details and references, including discussion of the rank deficient case and the singular value decomposition, which is an important factorization for analysing such problems, can be found in Golub and Van Loan (1983) and Hammarling (1985).

## 5. THE SQUARE ROOT FILTER

The use of the QR factorization is the basis of the so called square root filtering methods, in which Cholesky factors are recurred in place of the corresponding covariance or information matrices. We illustrate the approach by considering the square root filter corresponding to the Kalman filter of equations (2.3) - (2.6).

Let the Cholesky factorizations of $\hat{X}_{k-1}$, $U_k$ and $V_k$ be

$$\hat{X}_{k-1} = \hat{S}_{k-1}^T\hat{S}_{k-1}, \quad U_k = G_k^TG_k \quad \text{and} \quad V_k = H_k^TH_k \tag{5.1}$$

and, as before, let

$$\tilde{x}_k = A_k\hat{x}_{k-1}.$$

If we let $B_k$ denote

$$B_k = \begin{pmatrix} \hat{S}_{k-1} & A_k^T \\ & G_k \end{pmatrix}$$

then from the second of (2.3) we see that

$$\tilde{X}_k = B_k^TB_k$$

and hence if we let the QR factorization of $B_k$ be

$$B_k = Q_k \begin{pmatrix} \tilde{S}_k \\ 0 \end{pmatrix}, \tag{5.4}$$

where $Q_k$ is orthogonal, then

$$\tilde{X}_k = \tilde{S}_k^T \tilde{S}_k \tag{5.5}$$

and $\tilde{S}_k$ is the Cholesky factor of $\tilde{X}_k$.

Now let the Cholesky factorization of $\tilde{Y}_k$ be

$$\tilde{Y}_k = \tilde{T}_k^T \tilde{T}_k \tag{5.6}$$

and put

$$M_k = \tilde{T}_k^{-T} C_k \tilde{X}_k. \tag{5.7}$$

Then from the first of (2.4) we have

$$\tilde{T}_k^T \tilde{T}_k = \begin{pmatrix} H_k^T & (\tilde{S}_k C_k^T)^T \end{pmatrix} \begin{pmatrix} H_k \\ \tilde{S}_k C_k^T \end{pmatrix}$$

and from (2.5) we have

$$\tilde{S}_k^T \tilde{S}_k = \hat{S}_k^T \hat{S}_k + M_k^T M_k,$$

so that, if we let $F_k$ and $Z_k$ denote

$$F_k = \begin{pmatrix} H_k & 0 \\ \tilde{S}_k C_k^T & \tilde{S}_k \end{pmatrix} \quad \text{and} \quad Z_k = \begin{pmatrix} \tilde{T}_k & M_k \\ 0 & \hat{S}_k \end{pmatrix}, \tag{5.8}$$

we find that

$$F_k^T F_k = Z_k^T Z_k \tag{5.9}$$

and hence $Z_k$ is the Cholesky factor of $F_k^T F_k$. The QR factorization of $F_k$ must therefore be of the form

$$F_k = P_k \begin{pmatrix} \tilde{T}_k & M_k \\ 0 & \hat{S}_k \end{pmatrix}, \tag{5.10}$$

where $P_k$ is orthogonal. Finally, corresponding to (2.6) we now find that

$$\hat{x}_k = \tilde{x}_k + M_k^T d_k, \tag{5.11}$$

where $d_k$ is the solution of the lower triangular system of equations

$$\tilde{T}_k^T d_k = y_k - C_k \tilde{x}_k \tag{5.12}$$

In summary, for this square root filter, the computation proceeds as follows:

1) Compute $\tilde{x}_k$ from (5.2) and $\tilde{S}_k$ via the QR factorization of $B_k$. These give the time update corresponding to (2.3).

2) Form $F_k$ and compute $Z_k$ via the QR factorization of $F_k$, thus giving $\tilde{T}_k$, $M_k$ and $\hat{S}_k$.

3) Compute $\hat{x}_k$ from (5.12) and (5.11). The computation of $\hat{S}_k$ and $\hat{x}_k$ is the measurement update corresponding to (2.5) and (2.6) and the computation of $\tilde{T}_k$ and $M_k$ corresponds to (2.4).

$\tilde{S}_k$, $\tilde{T}_k$ and $\hat{S}_k$ are the Cholesky factors of $\tilde{X}_k$, $\tilde{Y}_k$ and $\hat{X}_k$ respectively.

We should note that the orthogonal matrices $Q_k$ and $P_k$ in the QR factorizations (5.4) and (5.10) are not required and, by using plane rotations (Golub and Van Loan, 1983, sections 3.4 and 12.6), it is possible to take advantage of the special form of $F_k$ in computing its QR factorization. We also note that $G_k$ and $H_k$ are often known, or can be obtained via QR factorizations, without the need to form $U_k$ and $V_k$ explicitly.

The square root filter seems first to have been suggested by Potter in 1964 and reviews of these methods can be found in Kaminski, Bryson and Schmidt, 1971 and in Bierman, 1977.

The method, as presented above, still requires that $\tilde{Y}_k$ be nonsingular since $\tilde{T}_k$ has been assumed to be non-singular and this is the normal assumption, but by considering a least squares solution of (5.12) this condition can be relaxed. The implementation of the square root filter using this approach is described by Kourouklis, 1977, in a, sadly, unpublished Master's thesis. There does seem to be some danger of loss of information in the computation of $M_k^T d_k$ in (5.11). We also have to explicilty form the products $\hat{S}_{k-1} A_k^T$ and $\tilde{S}_k C_k^T$ in $B_k$ and $F_k$ respectively. It is not clear whether or not this is numerically significant.

In the next section we present the Kalman filter as a standard least squares problem and this allows a clearer view of the numerical properties.

## 6. THE KALMAN FILTER AS A LEAST SQUARES PROBLEM

Let us assume that the initial estimate $\hat{x}_0$ is known so that

$$x_0 = \hat{x}_0 + u_0. \tag{6.1}$$

Then for t = 0, 1, ..., k equations (1.1) give

$$\begin{aligned} x_0 &= \hat{x}_0 + u_0, & y_0 &= C_0 x_0 + v_0, \\ x_1 &= A_1 x_0 + u_1, & y_1 &= C_1 x_1 + v_1, \\ &\vdots & &\vdots \\ x_k &= A_k x_{k-1} + u_k, & y_k &= C_k x_k + v_k \end{aligned}$$

which can be written as

$$b_k = B_k z_k + e_k \tag{6.2}$$

where

$$b_k = \begin{pmatrix} -\hat{x}_0 \\ y_0 \\ 0 \\ y_1 \\ 0 \\ \vdots \\ y_{k-1} \\ 0 \\ y_k \end{pmatrix}, \quad B_k = \begin{pmatrix} -I & 0 & 0 & \dots & 0 & 0 \\ C_0 & 0 & 0 & \dots & 0 & 0 \\ A_1 & -I & 0 & \dots & 0 & 0 \\ 0 & C_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & -I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_k & -I \\ 0 & 0 & 0 & \dots & 0 & C_k \end{pmatrix}, \quad z_k = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, \quad e_k = \begin{pmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \\ \vdots \\ u_k \\ v_k \end{pmatrix}.$$

If we make the assumption that

$$E(u_i u_j^T) = E(v_i v_j^T) = 0, \quad i \neq j, \quad E(u_i v_j^T) = 0 \tag{6.3}$$

and put

$$W_k = E(e_k e_k^T) \tag{6.4}$$

then from (1.1) and (6.2) we see that

$$W_k = \begin{pmatrix} U_0 & 0 & \dots & 0 & 0 \\ 0 & V_0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & U_k & 0 \\ 0 & 0 & \dots & 0 & V_k \end{pmatrix}. \tag{6.5}$$

Duncan and Horn (1972) showed that the Kalman filtering problem is equivalent to determining the component $\hat{x}_k$ of the solution $\hat{z}_k$ to the weighted least

squares problem

$$\text{minimize} \quad e_k^T W_k^{-1} e_k, \quad \text{where } b_k = B_k \hat{z}_k + e_k \tag{6.6}$$

The covariance filter of Kalman corresponds to updating the solution to the normal equations

$$B_k^T W_k^{-1} B_k \hat{z}_k = B_k^T W_k^{-1} b_k \tag{6.7}$$

from time t = k-1 to t = k.

If we let $R_k$ denote the Cholesky factor of $W_k$ so that, by (5.1),

$$W_k = R_k^T R_k, \quad R_k = \begin{pmatrix} G_0 & 0 & \dots & 0 & 0 \\ 0 & H_0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & G_k & 0 \\ 0 & 0 & \dots & 0 & H_k \end{pmatrix} \tag{6.8}$$

and let

$$r_k = R_k^{-T} e_k, \quad \text{so that } E(r_k r_k^T) = I, \tag{6.9}$$

then the weighted least squares problem (6.6) becomes

$$\text{minimize} \quad r_k^T r_k, \quad \text{where } R_k^{-T} b_k = R_k^{-T} B_k \hat{z}_k + r_k \tag{6.10}$$

and the covariance square root filter corresponds to updating the solution to this least squares problem using the QR factorization.

We briefly indicate one method of updating the solution of (6.10) from time t = k-1 to t = k. The matrix $R_{k-1}^{-T} B_{k-1}$ is given by

$$R_{k-1}^{-T} B_{k-1} = \begin{pmatrix} -G_0^{-T} & & & & & \\ H_0^{-T} C_0 & & & & & \\ G_1^{-T} A_1 & -G_1^{-T} & & & & \\ & H_1^{-T} C_1 & & & & \\ & G_2^{-T} A_2 & -G_2^{-T} & & & \\ & & H_2^{-T} C_2 & & & \\ & & & \vdots & & \\ & & & & H_{k-1}^{-T} C_{k-1} \end{pmatrix} \tag{6.11}$$

and the upper triangular factor, $\underline{R}_{k-1}$, of the QR factorization of $R_{k-1}^{-T}B_{k-1}$ has the form

$$\underline{R}_{k-1} = \begin{bmatrix} D_0 & E_1 & & & & \\ & D_1 & E_2 & & & \\ & & D_2 & & & \\ & & & \ddots & & \\ & & & & D_{k-2} & E_{k-1} \\ & & & & & \hat{D}_{k-1} \end{bmatrix}, \tag{6.12}$$

where the diagonal blocks $D_i$ and $\hat{D}_i$ are upper triangular. At time step $t = k$ we see that the only part of $\underline{R}_{k-1}$ that changes will be $\hat{D}_{k-1}$ and so we are concerned with the transformation from

$$\begin{bmatrix} \hat{D}_{k-1} & \\ G_k^{-T}A_k & -G_k^{-T} \\ & H_k^{-T}C_k \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} D_{k-1} & E_k \\ 0 & \hat{D}_k \\ & 0 \end{bmatrix},$$

which can be achieved in two QR factorization steps as indicated below

$$\begin{bmatrix} \hat{D}_{k-1} & \\ G_k^{-T}A_k & -G_k^{-T} \\ & H_k^{-T}C_k \end{bmatrix} \longrightarrow \begin{bmatrix} D_{k-1} & E_k \\ 0 & Z_k \\ & H_k^{-T}C_k \end{bmatrix} \longrightarrow \begin{bmatrix} D_k & E_k \\ & \hat{D}_k \\ & 0 \end{bmatrix}$$

Because $-G_k^{-T}$ is upper triangular, with judicious use of plane rotations we can make $Z_k$ upper triangular and then take advantage of the upper triangular form of $Z_k$ in computing $\hat{D}_k$.

Having performed the factorization and applied the transformations to the vector $b_k$, we then obtain $\hat{z}_k$, as in (4.10), as the solution of an upper triangular system of equations of the form

$$\underline{R}_{k-1}\hat{z}_k = \underline{c}_k. \tag{6.13}$$

If we partition $\underline{c}_k$ as

$$\underline{c}_k = \begin{bmatrix} \bar{c}_k \\ \hat{c}_k \end{bmatrix}, \tag{6.14}$$

then by considering the solution of (6.13) we see that $\hat{x}_k$ is simply the solution of the upper triangular equations

$$\hat{D}_k \hat{x}_k = \hat{c}_k. \tag{6.15}$$

Furthermore, since

$$E(\hat{z}_k \hat{z}_k^T) = (\underline{R}_{k-1}^T \underline{R}_{k-1})^{-1},$$

we similarly find that

$$\hat{X}_k = (\hat{D}_k^T \hat{D}_k)^{-1} = \hat{D}_k^{-1} \hat{D}_k^{-T}. \tag{6.16}$$

(6.16) shows that this particular method corresponds to an information square root filter.

This view of the problem seems to make the computational possibilities much clearer and it is more obvious how to apply existing numerical methods and error analyses to the problem, (see for example Lawson and Hanson, 1974). We can also readily see how to smooth and predict, as well as filter and by allowing a more general form for W we can easily relax conditions (6.3). This computational approach was first suggested by Paige and Saunders (1977) and full details and further discussion can be found in their paper.

The weighted least squares approach still relies on the existence of certain inverses, indeed problem (6.6) is not defined when $W_k$ is singular. In the next section we give a formulation that does not rely on non-singularity.

## 7. THE GENERALIZED LEAST SQUARES PROBLEM

Consider the weighted least squares problem

$$\text{minimize} \quad e^T W^{-1} e, \quad \text{where } b = Bz + e, \tag{7.1}$$

let F be a matrix such that

$$W = F^T F$$

and let r be a noise vector such that

$$F^T r = e.$$

As in (6.10), when F is non-singular, we can express (7.1) as

$$\text{minimize} \quad r^T r, \quad \text{where } F^{-T} b = F^{-T} B z + r \tag{7.2}$$

Paige (1978) noted that we can express this as the constrained least squares problem

$$\text{minimize} \quad r^T r, \quad \text{subject to } b = Bz + Fr. \tag{7.3}$$

While superficially this is a trivial rearrangement of (7.2), it has the vital difference of not now requiring non-singularity of F, and hence of W. Note

that if

$$E(rr^T) = I \quad \text{then} \quad E(ee^T) = W. \tag{7.4}$$

(7.3) is called the generalized linear least squares problem.

To indicate how (7.3) might be solved, let the QR factorization of B be

$$B = Q_B \begin{bmatrix} R_B \\ 0 \end{bmatrix}, \tag{7.5}$$

so that (7.3) becomes

$$\text{minimize} \quad r^T r, \quad \text{subject to } Q_B^T b = \begin{bmatrix} R_B \\ 0 \end{bmatrix} z + (FQ_B)^T r,$$

let the QR factorization of $FQ_B$ be

$$FQ_B = Q_F R_F \tag{7.6}$$

and put

$$Q_B^T b = c \equiv \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad Q_F^T r = p \equiv \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad R_F = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}. \tag{7.7}$$

(7.3) now becomes

$$\text{minimize} \quad p^T p, \quad \text{subject to } c = \begin{bmatrix} R_B \\ 0 \end{bmatrix} z + R_F^T p \tag{7.8}$$

so that

$$c_1 = R_B z + R_{11}^T p_1, \quad c_2 = R_{12}^T p_1 + R_{22}^T p_2$$

Hence we choose $p_1 = 0$ giving

$$R_{22}^T p_2 = c_2, \quad R_B z = c_1. \tag{7.9}$$

If $R_{22}$ and $R_B$ are non-singular then these triangular equations are readily solved for $p_2$ and $z$. Further details and analyses can be found in Paige (1979b) and Kourouklis and Paige (1981). More recently Paige (1985) has given an elegant analysis of the generalized least squares problem in terms of an important modern tool called the generalized singular value decomposition (Van Loan, 1976 and 1982; Paige and Saunders, 1981).

For the particular case of the Kalman filtering problem, much of the weighted least squares approach carries over to the generalized least squares approach, however there are still some questions to be answered. We can update the

matrix $B = B_k$ as described in the previous section, but it is not clear how best to update $F = F_k$. If we take the natural choice $F_k = R_k$ of (6.8) then it does not appear possible to prevent $R_F$ in (7.6) from being full in its upper triangular part. So there is still some work to be done here to investigate efficient, but stable, methods of updating from time step $t = k-1$ to $t = k$. Potential tools for achieving this are discussed in Paige (1978).

## 8. CONCLUSION

We have given a historical view of the numerical solution of the Kalman filtering problem, highlighting where the numerical difficulties occur and indicating which tools are useful in overcoming these difficulties. The generalized least squares approach is numerically the most reliable and both perturbation and rounding error analyses have been given by Paige (1979a). There remains some work to be done in applying this approach to the Kalman filtering problem, but it is to be hoped that the near future will see the emergence of reliable algorithms and quality software implementing these algorithms for this important problem.

## 9. REFERENCES

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