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A Note on Modifications to the Givens Plane Rotation

SVEN HAMMARLING
Enfield College, Middlesex Polytechnic

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It has recently been shown how to perform plane rotations without square roots and with a saving in multiplications. (See for example, Gentleman, 1973). The purpose of this note is to present the method in a simple but general manner which also suggests other modifications.

1. Standard Givens

Let

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & \dots & x_r \\ y_1 & y_2 & \dots & y_r \end{bmatrix}, \quad \mathbf{X}' = \begin{bmatrix} x_1' & x_2' & \dots & x_r' \\ 0 & y_2' & \dots & y_r' \end{bmatrix},$$

$$c = \cos\theta \quad \text{and} \quad s = \sin\theta.$$

The standard Givens plane rotation is of the form

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \mathbf{X} = \mathbf{X}'$$

where putting

$$d = (x_1^2 + y_1^2)^{\frac{1}{2}} \tag{1.1}$$

we have

$$c = x_1/d, \qquad s = y_1/d \tag{1.2}$$

and

$$x_1' = d \tag{1.3}$$

$$\left.\begin{aligned} x_i' &= cx_i + sy_i \\ y_i' &= -sx_i + cy_i \end{aligned}\right\} i = 2, 3, \dots, r. \tag{1.4}$$

Since each column of $\mathbf{X}'$ requires four multiplications this is referred to as a four multiplication transformation.

If we denote the computed x_i' and y_i' by $\bar{x}_i'$ and $\bar{y}_i'$ respectively then Wilkinson (1965) has shown that for t-digit floating point arithmetic

$$\begin{bmatrix} \bar{x}_i' - x_i' \\ \bar{y}_i' - y_i' \end{bmatrix} = \begin{bmatrix} c\alpha_1 & s\alpha_2 \\ -s\alpha_3 & c\alpha_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{1.5}$$

where $|\alpha_j| \leqslant 5 \,.\, 2^{-t}, j = 1, 2, 3, 4$. Since $|c|, |s| \leqslant 1$ this means that the transformation is stable.

2. Basic Modification

Put

$$\mathbf{X} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \dots u_r \\ v_1 & v_2 \dots v_r \end{bmatrix} = \mathbf{KU}$$

and

$$\mathbf{X}' = \begin{bmatrix} k_1' & 0 \\ 0 & k_2' \end{bmatrix} \begin{bmatrix} u_1' & u_2' & \dots & u_r' \\ 0 & v_2' & \dots & v_r' \end{bmatrix} = \mathbf{K}'\mathbf{U}'$$

so that

$$x_i = k_1 u_i, \qquad x_i' = k_1' u_i, \qquad y_i = k_2 v_i, \qquad y_i' = k_2' v_i'. \tag{2.1}$$

Instead of working directly with $\mathbf{X}$ and $\mathbf{X}'$ we use their factorizations $\mathbf{KU}$ and $\mathbf{K}'\mathbf{U}'$. Generally we regard $\mathbf{K}$ and $\mathbf{U}$ as given, so that we have freedom just in our choice of $\mathbf{K}'$.

Corresponding to equations (1.1) to (1.4) we have

$$d = (k_1^2 u_1^2 + k_2^2 v_1^2)^{\frac{1}{2}} \tag{2.2}$$

$$c = k_1 u_1 / d, \qquad s = k_2 v_1 / d \tag{2.3}$$

$$u_1' = d / k_1' \tag{2.4}$$

$$\left.\begin{aligned} u_i' &= (ck_1 u_i + sk_2 v_i)/k_1' \\ v_i' &= (-sk_1 u_i + ck_2 v_i)/k_2' \end{aligned}\right\} i = 2, 3, \dots, r. \tag{2.5}$$

If we denote by $\bar{a}$ the computed value of a and suppose that each of $\bar{d}$, $\bar{c}$, $\bar{s}$, $\bar{k}_1'$ and $\bar{k}_2'$ satisfy an equation of the form

$$\bar{a} = a(1+\varepsilon_a), \tag{2.6}$$

where ε_a is some modest multiple of 2^{-t}, then if we further put

$$\bar{x}_i' = \bar{k}_1' \bar{u}_i', \qquad \bar{y}_i' = \bar{k}_2' \bar{v}_i' \tag{2.7}$$

we have that

$$\left.\begin{aligned} \bar{x}_i' &= ck_1 u_i(1+\alpha_1) + sk_2 v_i(1+\alpha_2) = x_i' + cx_i\alpha_1 + sy_i\alpha_2 \\ \bar{y}_i' &= -sk_1 u_i(1+\alpha_3) + ck_2 v_i(1+\alpha_4) = y_i' - sx_i\alpha_3 + cy_i\alpha_4 \end{aligned}\right\} \tag{2.8}$$

where again each α_j, $j = 1, 2, 3, 4$ is a modest multiple of 2^{-t}. Equations (2.8) give

$$\begin{bmatrix} \bar{x}_i' - x_i' \\ \bar{y}_i' - y_i' \end{bmatrix} = \begin{bmatrix} c\alpha_1 & s\alpha_2 \\ -s\alpha_3 & c\alpha_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{2.9}$$

which is of the same form as equation (1.5) and means that any method based directly on equations (2.2) to (2.5) will be stable provided equation (2.6) holds for each of $\bar{d}$, $\bar{c}$, $\bar{s}$, $\bar{k}_1$ and $\bar{k}_2'$.

3. Choice of K'

One proposal (for example Gentleman, 1973) is to choose $\mathbf{K}'$ so that $u_1' = 1$ and $ck_2/k_2' = 1$. That is

$$k_1' = d, \qquad k_2' = ck_2. \tag{3.1}$$

If we assume that either previous rotations or an initial normalization have made

$$u_1 = 1$$

and we put

$$l_i = k_i^2, \qquad l_i' = (k_i')^2, \qquad i = 1, 2 \tag{3.2}$$

then this choice leads to the equations

$$l'_1 = l_1 + l_2 v_1^2, \qquad l'_2 = l_1 l_2 / l'_1 \tag{3.3}$$

$$\left.\begin{aligned} u'_i &= (l_1/l'_1)u_i + (l_2 v_1/l'_1)v_i \\ v'_i &= -v_1 u_i + v_i, \end{aligned}\right\} i = 2, 3, \ldots, r. \tag{3.4}$$

and since it is easy to show that equations (2.8) holds here, we have a stable three multiplication transformation without square roots. It has been noted (for example Gentleman, 1973) that

$$\begin{aligned} u'_i &= (l_1/l'_1)u_i + (l_2 v_1/l'_1)(v'_i + v_1 u_i) \\ &= (1/l'_1)(l_1 + l_2 v_1^2)u_i + (l_2 v_1/l'_1)v'_i \\ &= u_i + (l_2 v_1/l'_1)v'_i \end{aligned} \tag{3.5}$$

giving a two multiplication transformation. Unfortunately the first equation of (2.8) now no longer holds. Instead we find that

$$\bar{x}'_i - x'_i = c x_i \alpha_1 + s y_i \alpha_2 + \frac{x_i}{c}\alpha_5 \tag{3.6}$$

where α_5 is again a modest multiple of 2^{-t}. Here we can no longer guarantee stability since c could easily be small relative to x_i. A simple example is provided by

$$\mathbf{X} = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}$$

where ε is such that computationally $1+\varepsilon = 1$. An exact plane rotation gives

$$\mathbf{X}' = \begin{bmatrix} d & (\varepsilon+1)/d \\ 0 & (\varepsilon-1)/d \end{bmatrix}, \qquad d = (1+\varepsilon^2)^{\frac{1}{2}}$$

and computationally we are likely to get

$$\bar{\mathbf{X}}' = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

with low relative errors. After normalizing so that $u_1 = 1$ the above two multiplication transformation is likely to give

$$\bar{\mathbf{X}}' = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{\varepsilon^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{\varepsilon} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

with $\bar{x}'_2$ substantially in error.

Examination of equations (2.5) shows that there are six choices of $\mathbf{K}'$ that lead directly to two multiplication transformations. Typically, three such choices are

$$k'_1 = c k_1, \qquad k'_2 = c k_2 \tag{3.7}$$

or

$$k'_1 = c k_1, \qquad k'_2 = -s k_1 \tag{3.8}$$

or

$$k'_1 = c k_1, \qquad k'_2 = c^2 k_1 / s. \tag{3.9}$$

Using the notation of equation (3.2) the choice of (3.7) leads to the equations

$$l_1' = l_1\Big/\left(1+\frac{l_2v_1^2}{l_1u_1^2}\right), \qquad l_2' = l_2\Big/\left(1+\frac{l_2v_1^2}{l_1u_1^2}\right) \tag{3.10}$$

$$u_i' = u_i+\left(\frac{l_2v_1}{l_1u_1}\right)v_i, \qquad i = 1, 2, \ldots, r \tag{3.11}$$

$$v_i' = v_i-\left(\frac{v_1}{u_1}\right)u_i, \qquad i = 1, 2, \ldots, r. \tag{3.12}$$

The other choices lead to similar equations. It is easy to see that equation (2.8) holds in each case so that the transformations are stable. Since $|c| \leqslant 1$ there is clearly some danger of underflow in l_1' and l_2' when a sequence of rotations is involved. This can be avoided either by storing the exponent separately, by normalizing occasionally, or by performing row interchanges.

4. Applications

The two multiplication formulae make the method of Givens competitive with Householder in terms of speed and have distinct advantages for sparse systems. For example, the reduction of an $(n \times n)$ Hessenberg matrix to triangular form requires essentially $2n^2$ multiplications/divisions plus $(n-1)$ square roots by Householder's method, but only n^2 multiplications/divisions with the two multiplication transformation. This implies a considerable saving in methods such as the QR and QL methods for the eigenvalue problem (Wilkinson, 1971).

The method is easily extended to similarity transformations. If $\mathbf{A}$ is a given $(n \times n)$ matrix and $\mathbf{R}$ a plane rotation matrix then we use the factorizations

$$\mathbf{A} = \mathbf{KUK} \quad \text{and} \quad \mathbf{RAR}^T = \mathbf{K'U'K'},$$

where $\mathbf{K}$ and $\mathbf{K}'$ are diagonal and two straightforward applications of equations (2.5), (with $i = 1, 2, \ldots, r$) enable us to find $\mathbf{K}'$ and $\mathbf{U}'$. Full advantage may be taken of symmetry when it is present.

References

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